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On thermodiffusion of binary mixture in a horizontal channel at inhomogeneous heating the walls

Irina V. Stepanova*
Institute of Computational Modelling SB RAS
Krasnoyarsk, Russian Federation

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Abstract. The influence of the thermal load distributed unevenly at the walls of a long horizontal channel filled with a binary mixture on the convective flow is studied based on the constructed exact solution of the boundary value problem for the Oberbeck-Boussinesque equations. It is established that the obtained solution reflects the effect of thermal diffusion correctly, demonstrating the accumulation of a light component near a more heated wall. It is shown that an increase in the horizontal and vertical temperature gradient on the walls leads to the appearance of inhomogeneities in the temperature and concentration fields inside the layer.

Keywords: Oberbeck-Boussinesque equations, exact solution, binary mixture, convection, thermal diffusion separation.

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Introduction

It is well known that the occurring density gradient in liquids results in arising convection since the energy of gravity converts to the energy of motion. The density differences can be caused by heating/cooling of fluid or changing its concentration. It is necessary to note that description of convection in mixtures is much more complicated problem than the similar tasks related to homogeneous liquids. The interaction between thermal conductivity and diffusion, taking into account thermodiffusion effect can lead to complex relationships that need to be identified and studied. Analysis of regularities of convective motion can be provided by means of construction and interpretation of exact solutions of boundary value problems for differential equations describing the heat and mass transfer in liquid mixtures. Despite the intensive development of numerical methods for solving hydrodynamic problems, the construction of exact solutions does not lose its relevance. The solutions in closed formulas make it possible to carry out a complex analysis of parameters affecting the flow, revealing features of motions and dominant factors of heat and mass transfer.

One of the currently known solution related to the study of heat transfer in homogeneous liquid is mentioned in monograph of Ostroumov [1] and described in more detailed by Birikh [2]. It is the solution of classical Oberbeck-Boussinesque equations, where the velocity vector has the horizontal component only. In this case the temperature function depends on the longitudinal

*stepiv@icm.krasn.ru https://orcid.org/0000-0003-4255-2762

coordinate linearly. In the last decade the term "Birikh-type solution" refers to the entire class of solutions of equations of heat and mass transfer models, if it describes the flows in long horizontal channels, the walls of which are heated with respect to linear law in the longitudinal coordinate (see Fig. 1). The preprint [3] and paper [4] are devoted to the research of the Birikh-type solution in various problem statements and geometries of channel, and its nonstationary analogs as well. In [5] and in references therein the examples of utilization of similar to Birikh-type solution for study of evaporation from two liquids interface can be found.

The present paper concerns with using the Birikh-type solution for the modeling of thermal diffusion separation in a long horizontal channel filled by binary liquid mixture. This solution without final formulas is mentioned in [3]. Here, all expressions for velocity, temperature and pressure functions are obtained for one problem statement (if the both channel walls are heated with respect to linear law along the horizontal coordinate). The analysis of the influence of the thermal load of the walls is carried out. It is revealed that the main mechanism of generation of concentration inhomogeneities is the thermal diffusion effect. This is rather a rare situation, when at absence of the mentioned effect the field of concentration remains unchanged while temperature, pressure and velocity reconstruct significantly.

1. The problem statement

At the derivation of the mathematical model describing thermal diffusion separation in a binary liquid some essential assumptions are taken into account. They are the following:

- the flow is slow enough, its velocity is not great. Moreover, only the horizontal component of the velocity vector is not zero, the remaining components are neglected;
- heat and mass transfer is described with the help of Fourier's and Fick's laws respectively;
- density changes are considered only in the term corresponding to the buoyancy force, the dependence of density on concentration and temperature is linear.

Summing up the listed assumptions we write the equation of convection based on Navier–Stokes equations in the Oberbeck–Boussinesque approximation [6]. Furthermore, the balance relationships for temperature and concentration are taken into consideration. The governing system has the form

$$\begin{gathered}\nu u_{yy} = \frac{1}{\rho_0}\,p^*_x, \quad g(\beta_1 T + \beta_2 C) = \frac{1}{\rho_0}\,p^*_y,\\ uT_x = \chi(T_{xx} + T_{yy}),\\ uC_x = D(C_{xx} + C_{yy}) + D^\theta(T_{xx} + T_{yy}),\end{gathered} \tag{1}$$

where $u(y)$ is the horizontal component of the the velocity vector, the functions of temperature T, concentration C and the modified pressure up to hydrostatic term p^* depend on the both variables x and y, g is the gravity acceleration, ρ_0 is some reference value of the density of the liquid, ν, χ and D is the kinematic viscosity, heat diffusivity and diffusion coefficient respectively. $D^\theta < 0$ is the parameter of thermal diffusion, β_1 and β_2 are the coefficients of thermal and concentration expansion respectively. If the gradients of temperature and concentration are co-directional (oppositely directed) the thermal diffusion is called normal (abnormal) [7].

Let the values θ and L represent temperature and the scale of the width of the layer respectively. Using the following dimensionless variables

$$\hat{x} = \frac{x}{L}, \quad \hat{y} = \frac{y}{L}, \quad \hat{u} = \frac{\nu}{g\beta_1\theta L^2}\,u,$$
$$\hat{p^*} = \frac{1}{\rho_0 g\beta_1\theta L}\,p^*, \quad \hat{T} = \frac{1}{\theta}\,T, \quad \hat{C} = \frac{\beta_2}{\beta_1\theta}\,C, \tag{2}$$

we rewrite equations (1) in the nondimensional form

$$u_{yy} = p_x, \quad T + C = p_y,$$
$$\mathrm{Gr}\, u\, T_x = \frac{1}{\mathrm{Pr}}\,(T_{xx} + T_{yy}), \tag{3}$$
$$\mathrm{Gr}\, u\, C_x = \frac{1}{\mathrm{Sc}}\,[C_{xx} + C_{yy} - \psi(T_{xx} + T_{yy})],$$

the symbols hat and asterisk are omitted, $\mathrm{Gr} = g\beta_1\theta L^3/\nu^2$ is the Grashof number, $\mathrm{Pr} = \nu/\chi$ is the Prandtl number, $\mathrm{Sc} = \nu/D$ is the Schmidt number, $\psi = -\beta_2 D^\theta/(\beta_1 D)$ is the separation ratio. Further on the boundary conditions for system (3) are posed and discussed.

1.1. Boundary conditions

The scheme of flow is shown in Fig. 1. We consider the motion of binary mixture between two rigid walls heated with respect to linear law in the horizontal direction for the temperature distribution. In this case the boundary conditions for the temperature function are

$$T|_{y=0} = A_1 x + B_1, \qquad T|_{y=1} = A_2 x + B_2, \tag{4}$$

where A_i, B_i, $i = 1, 2$, are the given constants. They correspond to horizontal and vertical gradients of distribution of thermal load on the walls.

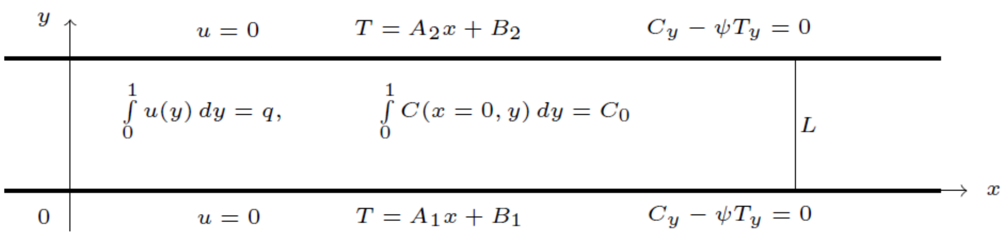

Fig. 1. Problem geometry and boundary conditions

For the function of velocity we have no-slip conditions. Therefore, the flow rate q through a cross section of the layer is assumed to be known. In such a way the following relations for the velocity function are fulfilled

$$u|_{y=0} = 0, \qquad u|_{y=1} = 0, \qquad \int_0^1 u\, dy = q. \tag{5}$$

For the function of concentration the conditions of the flux absence taking into account thermal diffusion effect are

$$\left(\frac{\partial C}{\partial y} - \psi\frac{\partial T}{\partial y}\right)\Big|_{y=0} = 0, \qquad \left(\frac{\partial C}{\partial y} - \psi\frac{\partial T}{\partial y}\right)\Big|_{y=1} = 0. \tag{6}$$

It is well known that when we deal with the boundary-value problem for an elliptic equation with the Neumann conditions (as the last equation in (3) with conditions (6)), then the question on closing the problem is open [8]. We need some additional condition for the exhaustive definition of all integrating constants. We use the condition for average concentration in the cross section $x = 0$, expressing by

$$\int_0^1 C(x=0,y)\,dy = C_0. \tag{7}$$

Equations (3) with conditions (4)–(7) are the closed boundary-value problem. Its solution is constructed and discussed further.

2. Exact solution of the posed boundary-value problem

One of the exact solution of system (3) with conditions (4)–(7) can be found if the functions of temperature and concentration depend on the variable x with respect to linear law as well as the temperature on the walls (see (4)). This assumption gives an opportunity to integrate equations (3) and retrace the solution satisfying to conditions (4)–(7) in the closed form. The functions of velocity, temperature, concentration and pressure are

$$u = \frac{(c_1+c_2)y^4}{24} + \frac{(c_3+c_4)y^3}{6} + \frac{c_5 y^2}{2} + c_6 y, \tag{8}$$

$$T = (c_1 y + c_3)x + \frac{\mathrm{Gr}\mathrm{Pr}\, y^3}{2}\Big(\frac{t_1 y^4}{21} + \frac{t_2 y^3}{15} + \frac{t_3 y^2}{10} + \frac{t_4 y}{6} + \frac{t_5}{3}\Big) + c_7 y + c_8, \tag{9}$$

$$C = (c_2 y + c_4)x + \frac{\mathrm{Gr}\, y^3}{2}\Big((\mathrm{Sc}\, k_1 + \psi\mathrm{Pr}\, t_1)\frac{y^4}{21} + (\mathrm{Sc}\, k_2 + \psi\mathrm{Pr}\, t_2)\frac{y^3}{15} + (\mathrm{Sc}\, k_3 + \psi\mathrm{Pr}\, t_3)\frac{y^2}{10} + {} + (\mathrm{Sc}\, k_4 + \psi\mathrm{Pr}\, t_4)\frac{y}{6} + \frac{\mathrm{Sc}\, k_5 + \psi\mathrm{Pr}\, t_5}{3}\Big) + c_9 y + c_{10}, \tag{10}$$

$$p = \Big[\frac{(c_1+c_2)y^2}{2} + (c_3+c_4)y + c_5\Big]x + \frac{\mathrm{Gr}\, y^4}{6}\Big((\mathrm{Sc}\, k_1 + (\psi+1)\mathrm{Pr}\, t_1)\frac{y^4}{56} + {} + (\mathrm{Sc}\, k_2 + (\psi+1)\mathrm{Pr}\, t_2)\frac{y^3}{35} + (\mathrm{Sc}\, k_3 + (\psi+1)\mathrm{Pr}\, t_3)\frac{y^2}{20} + (\mathrm{Sc}\, k_4 + (\psi+1)\mathrm{Pr}\, t_4)\frac{y}{10} + {} + \frac{\mathrm{Sc}\, k_5 + (\psi+1)\mathrm{Pr}\, t_5}{4}\Big) + (c_7+c_9)\frac{y^2}{2} + (c_8+c_{10})y + c_{11}. \tag{11}$$

The following notation for t_i and k_i, $i = 1,\ldots,5$, is used

$$\begin{aligned}
&t_1 = \frac{(c_1+c_2)c_1}{24}, \quad t_2 = \frac{(c_3+c_4)c_1}{6} + \frac{(c_1+c_2)c_3}{24},\\
&t_3 = \frac{c_5 c_1}{2} + \frac{(c_3+c_4)c_3}{6}, \quad t_4 = \frac{c_5 c_3}{2} + c_6 c_1, \quad t_5 = c_6 c_3,\\
&k_1 = \frac{(c_1+c_2)c_2}{24}, \quad k_2 = \frac{(c_3+c_4)c_2}{6} + \frac{(c_1+c_2)c_4}{24},\\
&k_3 = \frac{c_5 c_2}{2} + \frac{(c_3+c_4)c_4}{6}, \quad k_4 = \frac{c_5 c_4}{2} + c_6 c_2, \quad k_5 = c_6 c_4.
\end{aligned} \tag{12}$$

Constants c_i, $i = 1, \dots, 10$, are found by means of boundary conditions (4)–(7) and have the form

$$\begin{aligned}
&c_1 = A_2 - A_1, \quad c_2 = \psi(A_2 - A_1), \quad c_3 = A_1,\\
&c_4 = \frac{\psi(A_1 - A_2)}{2(\psi(A_1 - A_2) + 720q)}\left[A_1(\psi - 1) - A_2(\psi + 1) + 720q\right],\\
&c_5 = \frac{3\psi(A_1 - A_2)}{20} - \frac{(7A_1 + 3A_2)}{20} - \frac{c_4}{2} - 12q,\\
&c_6 = \frac{\psi(A_1 - A_2)}{30} + \frac{3A_1 + 2A_2}{60} + \frac{c_4}{12} + 6q,\\
&c_7 = B_2 - B_1 - \frac{\mathrm{GrPr}}{2}\left[\frac{t_1}{21} + \frac{t_2}{15} + \frac{t_3}{10} + \frac{t_4}{6} + \frac{t_5}{3}\right], \qquad c_8 = B_1, \qquad c_9 = \psi c_7,\\
&c_{10} = C_0 - \frac{\psi c_7}{2} - \frac{\mathrm{Gr}}{6}\left[\frac{\mathrm{Sc}\,k_1 + \psi\mathrm{Pr}\,t_1}{56} + \frac{\mathrm{Sc}\,k_2 + \psi\mathrm{Pr}\,t_2}{35} + \frac{\mathrm{Sc}\,k_3 + \psi\mathrm{Pr}\,t_3}{20} + \right.\\
&\qquad\qquad \left. + \frac{\mathrm{Sc}\,k_4 + \psi\mathrm{Pr}\,t_4}{10} + \frac{\mathrm{Sc}\,k_5 + \psi\mathrm{Pr}\,t_5}{4}\right].
\end{aligned} \tag{13}$$

The constant c_{11} remains unknown. It is a feature of all stationary problems in convection of liquids. The pressure function is defined up to one constant [9]. The obtained solution is the generalization of the Birikh solution for the case of studying a binary mixture flow.

3. Analysis of influence of temperature gradients on flow with the help of the constructed exact solution (8)-(13)

At the beginning of this study some essential remarks should be given.

1. If we put $\psi \equiv 0$, i.e. the thermodiffusion effect is excluded from the consideration, then simple calculations using formulas (10), (12) and (13) lead to $C \equiv C_0$. It means that the thermal diffusion effect is the main generator that builds up the concentration inhomogeneities. Further the case $\psi \equiv 0$ is left out of account.
2. Influence of flow rate q, gravity intensity g and layer thickness L on the flow was studied in [11]. It was concluded that when the separation ratio $\psi \neq 0$, the heterogeneity of the concentration field was more pronounced at the large layer thickness, zero flow rate and in the conditions of hypergravity action (a hundred times more than Earth gravity $g = 9.8\,\mathrm{m/s^2}$).
3. If $A_1 = A_2$, then the constants c_1, c_2 and c_4 vanish in formulas (13). It leads to the function of concentration (10) does not depend on x, and factor outside the x in the function of temperature (9) does not contain the variable y. It makes difficult the physical interpretation of the constructed solution. Further the case $A_1 = A_2$ is out of consideration.

Influence of temperature gradients A_i, B_i, $i = 1, 2$, is considered further in this paper. The following geometrical and physical parameters of the flow are used for analysis of action of the horizontal and vertical temperature gradients on the flow:

- The layer width $L = 0.003$ m, the gravity $g = 9.81$ m/s^2, the characteristics temperature difference $\theta = 20$ °C, the flow rate $Q = 10^{-5}$ kg/$(m \cdot s)$ ($q = Q\nu\rho_0\mathrm{Gr} = = 0.516 \cdot 10^{-5}$ is the nondimensional analog of Q), the nondimensional variable x varies in the interval (0, 20).

- The Prandtl number Pr=27.83, the Schmidt number Sc=5234, the separation ratio ψ=0.24 and the Grashof number Gr=957.55 correspond to physical parameters for ethanol-water mixture with the concentration of the ethanol $C_0 = 0.7$. The data for the calculations of these constants are taken from [10] and from the previous point of this list.

- The distribution for the temperature at the upper wall does not change. It is

$$T|_{y=1} = A_2 x + B_2 = 0.00015x + 0.5$$

  in the nondimensional variables for all considered configurations. The temperature distribution on the lower wall is

$$T|_{y=0} = A_1 x + B_1 = (A_2 + \Delta A)x + B_2 + \Delta B,$$

  where ΔA and ΔB are used further for the estimate of influence of horizontal and vertical temperature gradients on the thermal diffusion process.

The first point of the analysis is the investigation of action of changing ΔA on convection in the channel. We use $B_1 = 1$, the values of ΔA are given in the first column of Tab. 1. There are changes of the concentration and temperature functions in the second and third columns of Tab. 1 respectively. The values of ΔC and ΔT are found as difference between maximum and minimum values of the corresponding functions in the computational domain ($y \in [0, 1]$, $x \in [0, 20]$). It can be observed that intensification of the thermal load of the lower wall leads to increasing of values ΔC and ΔT. It should be emphasize that small variations of ΔA correspond to small variations of temperature and concentration differences.

Table 1. Estimate of influence of horizontal temperature gradient on changes of concentration and temperature

ΔA	ΔC	ΔT
0.00012	0.12561	0.527
0.001575	0.12751	0.534
0.00195	0.12952	0.542

The fields of ethanol concentration and mixture temperature are shown in Fig. 2 *b,c* for the values of $\Delta A = 0.0012$. It can be seen that the ethanol accumulates close to more heated lower wall. It is in a good agreement with the velocity profiles (Fig. 2 *a*), where increasing of the velocity values near the lower wall is observed. The solid, doted and dashed curves in Fig. 2 *a* correspond to different values of ΔA from Tab. 1. The growth of thermal gradient in the horizontal direction results in enrichment of more heated region by the light component. It explains increase of the velocity of the convective motion with respect to intensification of thermal load.

The second point of the analysis is the effect of vertical temperature gradient on the motion. As in the previous case the table with changes of ΔB, ΔC and ΔT is drawn up (see Tab. 2). The value of A_1 equals to 0.00135 for all ΔB in this study.

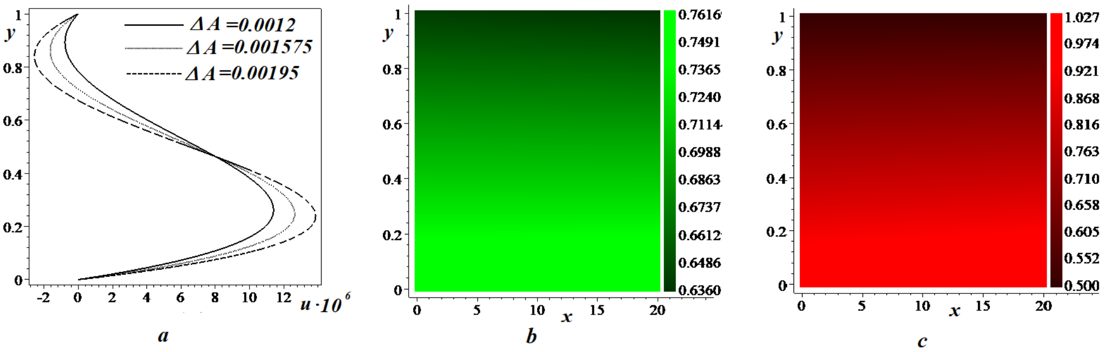

Fig. 2. The velocity profile (a) for different values of ΔA: the solid curve corresponds to $\Delta A = 0.0012$, the dot curve is for $\Delta A = 0.001575$ and the dash curve is for $\Delta A = 0.00195$. Fields of concentration (b) and temperature (c) for $\Delta A = 0.0012$, $B_1 = 1$

Table 2. Estimate of influence of vertical temperature gradient on changes of concentration and temperature

ΔB	ΔC	ΔT
0.25	0.30	1.253
0.5	0.36	1.503
0.75	0.42	1.753

It is necessary to note that the values of ΔB do not influence velocity. It can be confirmed with the help of expression (8) and corresponding constants from (13). There are no B_i, $i = 1, 2$, in these formulas. Thus, the velocity remains the same for every value of ΔB, the velocity profile can be seen in Fig. 2 a for $A_1 = 0.00135$. The profiles of concentration and temperature at $x = 10$ are given in Fig. 3 a,b for different values of ΔB given in the first column of Tab. 2. As in the table as by means of figure it can be followed that the growth of the vertical temperature gradient leads to increasing of concentration and temperature differences. Therefore, the profiles of T and C coincide, the ethanol tends to the lower wall $y = 0$, where the temperature is higher. This configuration corresponds to normal thermodiffusion effect [10].

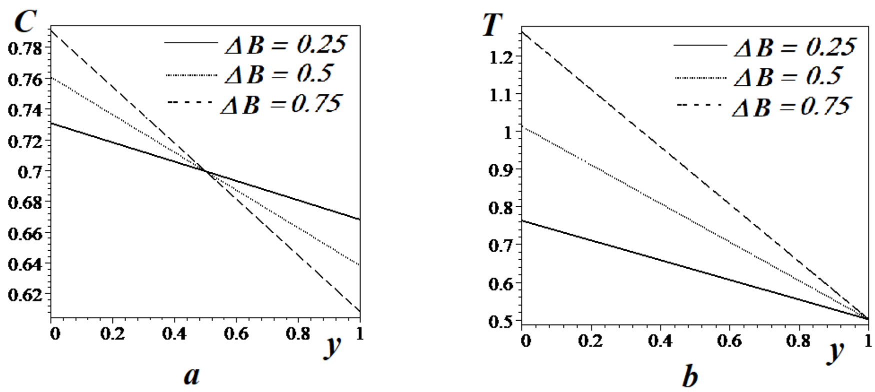

Fig. 3. The concentration profile (a) and temperature profile (b) in section $x = 10$ for different values of ΔB: the solid curve corresponds to $\Delta B = 0.25$, the dot curve is for $\Delta B = 0.5$ and the dash curve is for $\Delta B = 0.75$

Conclusion

The study presented in the paper is the final step of the exhaustive investigation of exact solution of the Birikh type describing convective motion in a binary liquid mixture in a long flat horizontal channel. Based on the constructed solution, it is possible to reveal the area of parameters where the solution does not exist from the point of view of mathematics (parameters that turn the denominator of the formula for c_4 in (13) to zero), or has no physical meaning (the concentration goes beyond the interval (0,1)). This conclusion emphasizes worth of exact solutions, because the solution constructed in the closed formulas allows simultaneous analysis of all factors influencing the flow.

The action of the thermal load linearly depending on the longitudinal coordinate at the channel walls has been analyzed. It is shown that the transverse temperature gradient does not affect the fluid velocity, but significantly effects the appearance of temperature and concentration inhomogeneities. At the same time an increase in the longitudinal temperature gradient on the walls results in an intensification of the velocity and has little effect on the redistribution of temperature and concentration in the layer. The presented paper together with another paper of the author [11], is a completed study of the effect of all variable system parameters (liquid flow rate, layer thickness, gravity action and thermal load on the walls) on the thermal diffusion separation of a binary mixture in a channel in stationary case. Further development of the study is the stability analysis of the obtained solution and the solution of corresponding nonstationary problem. The first results in investigation of the nonstationary problem can be found in [12].

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О термодиффузии бинарной смеси в горизонтальном канале с неоднородным нагревом стенок

Ирина В. Степанова
Институт вычислительного моделирования СО РАН
Красноярск, Российская Федерация

Аннотация. На основе построенного точного решения краевой задачи для уравнений Обербека-Буссинеска исследовано влияние тепловой нагрузки, распределенной неравномерно на стенках протяженного горизонтального канала, заполненного бинарной смесью, на конвективное течение. Установлено, что полученное решение корректно моделирует эффект термодиффузии в смеси вода-этанол, показывая скопление легкого по плотности компонента возле более нагретой стенки. Показано, что увеличение горизонтального и вертикального градиента температуры на стенках влечет появление неоднородностей полей температуры и концентрации внутри слоя.

Ключевые слова: уравнения Обербека-Буссинеска, точное решение, бинарная смесь, конвекция, термодиффузионное разделение.